# Asymptotic effects of boundary perturbations in excitable systems

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## Abstract

A Neumann problem in the strip for the Fitzhugh Nagumo system is considered. The transformation in a non linear integral equation permits to deduce a priori estimates for the solution. A complete asymptotic analysis shows that for large $t$ the effects of the initial data vanish while the effects of boundary disturbances $\varphi_1(t)$, $\varphi_2(t)$ depend on the properties of the data. When $\varphi_1$, $\varphi_2$ are convergent for large $t$, the solution is everywhere bounded; when $\dot{\varphi_i} \in L^1(0, \infty)(i = 1, 2)$ too, the effects are vanishing.

## 1 Introduction

Aim of the paper is the asymptotic analysis of the solution of the Fitzhugh Nagumo system (FHN) for a strip ploblem with Neumann conditions. Some applications are related to the theory of excitable systems; in particular the cases of pacemakers [11] and when two species reaction-diffusion systems is governed by flux boundary condition [16]. Moreover, Neumann conditions are applied also in the distibuted FHN system. [17].Several aspects concerning the FHN model are discussed in previous paper [4, 9, 10]. Moreover, owing to the equivalence between the FHN model and the equation of superconductivity, other applications have been analyzed. [3] - [7], [19, 20].

The present paper analyzes a transformation of the FHN model in a suitable non linear integral-equation (see 3.14) whose kernel is a Green function which has numerous basic properties typical of the diffusion equation. Those properties imply a priori estimates and so theorems on behaviour of the solution for large t can be obtained.

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## 2    Statement of the problem

Let $u(x,t)$ be a trasmembrane potential and let $v(x,t)$ be a variable associated with the contributions to the membrane current from sodium , potassium and other ions. The well known FHN system [11, 12, 15, 16, 19, 20] is

(2.1)
$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - v + f(u) \\[2mm] \dfrac{\partial v}{\partial t} = b u - \beta v \end{cases}$$

where $\varepsilon > 0$ is a diffusion coefficient related to the axial current in the axon, while $b$ and $\beta$ are positive constants that characterize the model's kinetic. Further

(2.2)
$$f(u) = u (a - u) (u - 1) \qquad (0 < a < 1).$$

Assuming $T$ as an arbitrary positive value, a typical example of problems which takes into account either initial perturbations and boundary perturbations is defined in

$$\Omega \equiv \{ (x,t) : 0 \le x \le L ; \ 0 < t < T \}$$

by

(2.3)
$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x)$$

with the Neumann conditions

(2.4)
$$u_x(0,t) = \varphi_1(t) \qquad u_x(L,t) = \varphi_2(t).$$

It can be easiy verified (see,f.i [4, 9]) that the problem can be analyzed by means of an integral differential problem with a single unknown function $u(x,t)$. In fact, if $F$ denotes the function:

(2.5)
$$F(x,t,u) = u^2 ( a + 1 - u ) - v_0 \, e^{-\beta t}$$

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by (2.1) - (2.2) one has

$$(2.6) \qquad u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)} u(x,\tau) \, d\tau = F(x,t,u) \qquad (x,t) \in \Omega$$

with $u$ that has to satisfy the initial - boundary conditions $(2.3)_1$, (2.4).

As soon as $u(x,t)$ is determined, the $v(x,t)$ component will be given by

$$(2.7) \qquad v(x,t) = v_0 \, e^{-\beta t} + b \int_0^t e^{-\beta(t-\tau)} u(x,\tau) \, d\tau,$$

where $v_0$ is defined in $(2.3)_2$.

When source term $F$ in (2.6) is a prefixed function depending only on $x$ and $t$, then the initial-boundary problem (2.6), $(2.3)_1$, (2.4) is linear and it can be solved explicitly by means of the Laplace transform. Moreover, when $F$ depends on the unknown $u(x,t)$ too, then by (2.6) one obtains an integral equation useful to study the differential problem.

# 3  Previous results

The fundamental solution $K(x,t)$ of the parabolic operator defined by (2.6) has been already determined explictly in [9] and is given by

$$(3.8) \quad K(r,t) = \frac{1}{2\sqrt{\pi\varepsilon}} \left[ \frac{e^{-\frac{r^2}{4t}-at}}{\sqrt{t}} - b \int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}} \, e^{-\beta(t-y)} J_1(2\sqrt{by(t-y)} \,) dy \right]$$

where $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind and order $n$. Moreover, one has [9]:

**Teorema 3.1.** *For all $t > 0$, the Laplace transform of $K(r,t)$ with respect to $t$ converges absolutely in the half-plane $\Re s > max(-a, -\beta)$ and it results:*

$$(3.9) \qquad \hat{K}(r,s) = \int_0^\infty e^{-st} K(r,t) \, dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon}\,\sigma}$$

*with $\sigma^2 = s + a + \frac{b}{s+\beta}$.*

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Let us now consider the following Laplace transforms with respect to $t$:

$$\hat{u}(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt\ , \quad \hat{F}(x,s) = \int_0^\infty e^{-st}\, F\,[\,x,t,u(x,t)\,]\, dt\ ,$$

and let $\hat{\varphi}_1(s),\ \hat{\varphi}_2(s)$ be the $L$ transforms of the data $\varphi_i(t)$ $(i=1,2)$.

Then the Laplace transform of the problem (2.6), (2.3)$_1$, (2.4) is formally given by:

(3.10)
$$\begin{cases} \hat{u}_{xx} \ -\ \dfrac{\sigma^2}{\varepsilon}\ \hat{u} \ =\ -\dfrac{1}{\varepsilon}\ [\ \hat{F}(x,s,\hat{u}(x,s)) + u_0(x)\ ] \\[2ex] \hat{u}_x(0,s) \ =\ \hat{\varphi}_1\,(s) \qquad \hat{u}_x(L,s) \ =\ \hat{\varphi}_2\,(s). \end{cases}$$

If one introduces the following theta function

(3.11)
$$\hat{\theta}\,(\,y,\sigma\,) \ =$$
$$\frac{1}{2\,\sqrt{\varepsilon}\ \sigma}\left\{ e^{-\frac{y}{\sqrt{\varepsilon}}\,\sigma} + \sum_{n=1}^\infty \left[ e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\,\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\,\sigma} \right] \right\}$$
$$= \frac{\cosh\,[\,\sigma/\sqrt{\varepsilon}\ (L-y)\,]}{2\,\sqrt{\varepsilon}\ \sigma\ \sinh\,(\,\sigma/\sqrt{\varepsilon}\ L\,)} \ =$$

then, by (3.10) and (3.11) one deduces:

(3.12)
$$\hat{u}(x,s) = \int_0^L [\,\hat{\theta}\,(\,|x-\xi|,\,s\,) + \hat{\theta}\,(\,|x+\xi|,\,s\,)\,]\,[\,u_0(\xi) + \hat{F}(\xi,s,\hat{u}(x,s)\,]\,d\xi$$

$$-\ \ 2\,\varepsilon\,\hat{\varphi}_1\,(s)\,\hat{\theta}(x,s) + 2\,\varepsilon\,\hat{\varphi}_2\,(s)\,\hat{\theta}\,(L-x,s).$$

Owing to dependence of source term $F$ on the unknown, obviously all this is purely formal. However, if one puts

(3.13)
$$\begin{cases} \theta(x,t) \ =\ \displaystyle\sum_{n=-\infty}^\infty\ K(x+2nL,t) \\[2ex] G(x,\xi,t) \ =\ \theta\,(\,|x-\xi|,\,t\,) + \theta\,(\,|x+\xi|,\,t\,) \end{cases}$$

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by (3.12) one deduces [4]:

$$u(x, t) =$$

$$\int_0^L G(x, \xi, t) \, u_0(\xi) \, d\xi \;-\; 2\,\varepsilon \int_0^t \theta\,(x, t-\tau) \; \varphi_1(\tau) \; d\tau$$

(3.14)

$$+ \; 2\,\varepsilon \int_0^t \theta\,(L-x, t-\tau) \; \varphi_2(\tau) \; d\tau$$

$$+ \; \int_0^t d\tau \int_0^L G(x, \xi, t-\tau) \;\; F\,[\,\xi, \tau, u\,(\,\xi, \tau\,)\,] \; d\xi.$$

which represents an integral equation for the unknown $u(x,t)$.

# 4 Basic estimates for the kernels $K(x,t)$ and $\theta(x,t)$

The behaviour for large $t$ of the terms depending on the initial data and the source $F$ has been already analyzed in [4] [5]. Now the effects of the boundary perturbations $\varphi_1$, $\varphi_2$ will be estimated. For this an appropriate analysis of the kernels $K(x,t)$ and $\theta(x,t)$ will be considered.

As for $K(x,t)$, in [9] has been proved that

(4.15)
$$|K| \;\leq\; \frac{e^{-\frac{r^2}{4t}}}{2\,\sqrt{\pi \varepsilon t}} \, [\, e^{-at} \,+\, bt\,E(t)\,]$$

where

(4.16)
$$E(t) \;=\; \frac{e^{-\beta t} \,-\, e^{-at}}{a-\beta} \;>\; 0\,.$$

Further, it results too:

(4.17)
$$\int_{\Re} |K(x-\xi, t)| \, d\xi \;\leq\; e^{-at} \,+\, \sqrt{b}\,\pi\, t\, e^{-\omega t}$$

(4.18)
$$\int_0^t d\tau \int_{\Re} |K(x-\xi, t)| \, d\xi \;\leq\; \beta_0.$$

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with

$$(4.19) \qquad \omega = min\,(a, \beta) \qquad \beta_0 = \frac{1}{a} + \pi\sqrt{b}\,\frac{a + \beta}{2(a\beta)^{3/2}}.$$

Now, if $\Gamma(x)$ is the gamma function and $\zeta(x)$ the Riemann's Zeta function, let

$$(4.20) \qquad C_0 = \frac{1}{2\sqrt{\varepsilon\,\omega}} + \frac{b\,\Gamma(3/2)\,\omega^{-3/2}}{2\sqrt{\pi\,\varepsilon}\,|a - \beta|}\left[1 + \frac{C}{b}|a - \beta| + \frac{3\,C}{2\,\omega}\right]$$

with $C = 2\varepsilon\,\zeta(2)/(\,eL^2\,)$.

Then, one has the following theorem:

**Teorema 4.2.** *The $\theta(x, t)$ function defined in* $(3.13)_1$ *satisfies the following inequalities:*

$$(4.21) \qquad \int_0^L |\theta(|x - \xi|, t)|\ d\xi \le (1 + \sqrt{b}\,\pi\,t)\,e^{-\omega\,t}$$

$$(4.22) \qquad \int_0^t d\tau \int_0^L |\theta(|x - \xi|, t)|\ d\xi \le \beta_0 \qquad .$$

*Furthermore, it results:*

$$(4.23) \qquad \lim_{t\to\infty}\,\theta(x, t)\,=\,0; \qquad \int_0^\infty |\theta(x, \tau)|\ d\tau\,\le\,C_0,$$

*and*

$$(4.24) \qquad \lim_{t\to\infty}\int_0^t \theta(x, \tau)\ d\tau\,=\,\frac{1}{2\,\varepsilon\,\sigma_0}\,\frac{\cosh\sigma_0\,(x - L)}{\sinh\,(\sigma_0\,L)}$$

*where* $\sigma_0 = \sqrt{\left(a + \dfrac{b}{\beta}\right)\dfrac{1}{\varepsilon}}.$

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*Proof.* : We observ that properties of $K(x,t)$ imply that:

$$\int_0^L |\theta(|x-\xi|,\, t)|\, d\xi \;\leq\; \sum_{n=-\infty}^{\infty} \int_0^L |K(|x-\xi+2nL|\,t)|\, d\xi$$

(4.25)

$$= \sum_{n=-\infty}^{\infty} \int_{x+(2n-1)L}^{x+2nL} |K(y,\,t)|\, dy \;\leq\; \int_{\Re} |K(y,t)|\, dy$$

and so (4.21) and (4.22) follow by (4.17) and (4.18).

Moreover , it results

$$(4.26) \qquad \sum_{n=-\infty}^{\infty} e^{-\frac{(x+2nL)^2}{4\varepsilon t}} \;\leq\; 1 + \frac{2\,t\,\varepsilon}{e\,L^2}\, \zeta(2)$$

and (4.15)implies:

$$(4.27) \qquad |\theta(x,t)| \;=\; \frac{1+C\,t}{2\sqrt{\pi\,\varepsilon\,t}}\,[\,e^{-a\,t} + b\,t\,E(t)\,],$$

consequently one obtains $(4.23)_1$ while considered that

$$(4.28) \quad \int_0^\infty t^\mu e^{-\omega t} dt = \frac{\Gamma(\mu+1)}{\omega^{\mu+1}} \quad \begin{array}{l} re(\mu) > -1 \\ re(\omega) > 0 \end{array} \;\; ; \;\; \int_0^\infty \frac{e^{-at}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{a}} \quad a > 0,$$

by means of (4.15), $(4.23)_2$ can be deduced.

Further as:

$$(4.29) \qquad \lim_{t\to\infty} \int_0^t \theta(x,\tau)\, d\tau \;=\; \lim_{s\to 0} \hat{\theta}(x,s) \qquad \Re e\, s > max(-a,\, -\beta\,),$$

by (3.11), one achieves(4.24).

$\square$

# 5 Asymptotic effects of the boundary data

In the following we will have to refer to a known theorem on asymptotic behaviour of convolutions. ( [1],p 66).

*Let $h(t)$ and $g(t)$ be two continuous functions on $[0, \infty[$. If they satisfy the following hypotheses*

(5.30) $$\exists \lim_{t \to \infty} h(t) = h(\infty) \qquad \exists \lim_{t \to \infty} g(t) = g(\infty),$$

(5.31) $$\dot{g}(t) \in L_1[0, \infty),$$

*then, it results:*

(5.32) $$\lim_{t \to \infty} \int_o^t h(t - \tau)\, \dot{g}(\tau)\, d\tau = h(\infty)\, [\, g(\infty) - g(0)\, ].$$

According to this, it is possible to state:

**Teorema 5.3.** *Let $\varphi_i$ $(i = 1, 2)$ be two continuous functions which converge for $t \to \infty$. In this case one has:*

(5.33) $$\lim_{t \to \infty} \int_0^t \theta\,(x, \tau)\ \varphi_i\,(t - \tau)\ d\tau = \varphi_{i,\infty}\ \frac{1}{2\,\varepsilon\,\sigma_0}\ \frac{\cosh \sigma_0\,(x - L)}{\sinh\ \sigma_0\, L}$$

*where $\sigma_0 = \sqrt{\left( a\ +\ \dfrac{b}{\beta} \right) \dfrac{1}{\varepsilon}}$.*

*Proof.* Let apply (5.32) with $g = \int_0^t \theta(x, \tau)d\tau$ and $f = \varphi_i$ $(i = 1, 2)$. Then, (5.33) follows by $(4.23)_2$ and (4.24). $\qquad\square$

**Teorema 5.4.** *When the data $\varphi_i$ $(i = 1, 2)$ verify conditions (5.30) (5.31), it results:*

$$(5.34) \qquad \lim_{t \to \infty} [\theta(x, t) * \varphi_i(t)] = 0 \qquad (i = 1, 2)$$

*Proof.* It sufficies to put $h = \theta(x, t)$ and $g = \varphi_i$ and to apply $(4.23)_1$.   □

# 6   Asymptotic behaviour of the FHN solution

Let us denote with $f_1 * f_2$ the convolution

$$f_1(\cdot, t) * f_2(\cdot, t) = \int_0^t f_1(\cdot, t) \, f_2 \, (\cdot, t - \tau) \, d\tau$$

and let $N(x, t)$ be the following known function depending on the data $(u_0, v_0, \varphi_1, \varphi_2)$

$$N(x, t) =$$
$$- 2\varepsilon \, \varphi_1(t) * \theta(x, t) + 2\varepsilon \, \varphi_2(t) * \theta(L - x, t)$$

(6.35)

$$+ \int_0^L u_0(\xi) \, G(x, \xi, t) \, d\xi - e^{-\beta t} * \int_0^L v_0(\xi) \, G(x, \xi, t) \, d\xi.$$

Owing to (2.5), (2.7) and (3.14),the solution related to the initial boundary FHN system 2.1-2.4 is given by [4]:

$$(6.36) \quad u(x, t) = \int_0^L G(x, \xi, t - \tau) * \{ u^2(\xi, \tau)[a + 1 - u(\xi, \tau)]\} \, d\xi + N$$

(6.37)
$$v(x, t) = v_0 \, e^{-\beta t} + b \, e^{-\beta t} * N(x, t)$$

$$+ b \, e^{-\beta t} * \int_0^L G(x, \xi, t - \tau) * \{ u^2(\xi, \tau)[a + 1 - u(\xi, \tau)]\} \, d\xi$$

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These formulae represent two integral equations for $u$ and $v$. By means of the estimates deduced in sec.4 it is possible to apply the fixed point theorem in order to obtain existence and uniqueness results [2, 4, 8]. When the Nagumo polinomial (2.5) is approximated by means of its linear part, then (6.36) (6.37) give the explicit solution of the problem.

As for the analysis and the stability of solutions of nonlinear binary reaction - diffusion systems of PDE's, as well as the existence of global compact attractors, there exists a large bibliography . (see e. g. [10, 12–14, 18]. Moreover, as it is well known,the (FHN) system admits arbitrary large invariant rectangles $\Sigma$ containing $(0,0)$ so that the solution $(u, v)$, for all times $t > 0$, lies in the interior of $\Sigma$ when the initial data $(u_o, v_o)$ belong to $\Sigma$. [21]
So, letting

$$\| F \| = \sup_{\Omega_T} \mid u^2 \, (a + 1 - u) \mid,$$

$$\| u_0 \| = \sup_{\Omega_T} \mid u_0 \, (x, ) \mid; \qquad \| v_0 \| = \sup_{\Omega_T} \mid v \, (x) \mid$$

one has:

**Teorema 6.5.** *For regular solution $(u, v)$ of the (FHN) model, when the boundary conditions are homogeneous, ( $\varphi_1 = \varphi_2 = 0$ ), the following estimates hold:*

(6.38)
$$\begin{cases} |u| \leq 2 \, [\, \|u_0\| \, (1 + \pi \sqrt{b} \, t) \, e^{-\omega t} \, + \, \|v_0\| \, E(t) \, + \, \beta_0 \, \|F\| \, ] \\[4mm] |v| \leq \|v_0\| \, e^{-\beta t} \, + \, 2 \, [\, b \, (\, \|u_0\| \, + \, t \, \|v_0\| \, ) \, E(t) \, + \, \frac{b}{a\beta} \, \|F\| \, ] \end{cases}$$

For boundary data different from zero,the asymptotic behaviour of the solution $(u, v)$ of FHN system is established by theorems 5.3 and 5.4.

**In conclusion**. When $t$ tends to infinity, the effect due to the initial disturbances $(u_0, v_0)$ vanishes while the effect of the non linear source is bounded for all $t$. Moreover, also the effects determined by boundary disturbance $\varphi_1, \varphi_2$ are vanishing in the hypotheses $(b)$. Otherwise, they are always bounded.

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